# Efficient Algorithm for Numerical Calculation of Trigonometric Hyperbolic Functions

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In the present work efficient algorithm for numerical calculation of sine and cosine hyperbolic functions is developed. Sine and cosine hyperbolic functions are defined by the following formulas:

$$\cosh(x)=\frac{e^x+e^{-x}}{2},\quad \sinh(x)=\frac{e^x-e^{-x}}{2}. \qquad (1)$$

There is constructed recurrent two layer scheme for approximate calculation of (1) sine and cosine hyperbolic matrix functions. Main idea of the algorithm is to calculate $\cosh(x_0A)$ and $A^{-1}\sinh(x_0A)$ for small values of $x_0=l/2^n$ and then reconstruct $\cosh(lA)$ and $A^{-1}\sinh(lA)$ using the following well-known double angle trigonometric formulas:

$$\cosh(lA)=2\cosh^2(Al/2)-I\,, \qquad (2)$$

$$A^{-1}\sinh(lA)=(lA/2)^{-1}\sinh(lA/2)\cosh(lA/2). \qquad (3)$$

Let us introduce the following notations:

$$U(x)=\cosh(xA)=I+\frac{x^2}{2!}A^2+\frac{x^4}{4!}A^4+\ldots,\qquad x\in[0,l]. \qquad (4)$$

$$V(x)=(xA)^{-1}\sinh(xA)=I+\frac{x^2}{3!}A^2+\frac{x^4}{5!}A^4+\ldots,\qquad x\in[0,l] \qquad (5)$$

Obviously, according to the (2) and (3), $U(x)$ and $V(x)$ satisfy the following recurrent relations:

$$U(x_{k+1})=2U^2(x_k)-I,\quad V(x_{k+1})=V(x_k)U(x_k), \qquad (6)$$

where $x_k=2^k/2^n l,\quad k=0,1,\ldots,n$. For numerical calculation of (6), we need initial values $U(x_0)$ and $V(x_0)$. Let us calculate approximate values of these matrices by the following formulas:

$$U_0=P_0(x_0A)=I+\frac{x_0^2}{2!}A^2+\frac{x_0^4}{4!}A^4+\ldots+\frac{x_0^{2p}}{(2p)!}A^{2p}, \qquad (7)$$

$$V_0=I+\frac{x_0^2}{3!}A^2+\frac{x_0^4}{5!}A^4+\ldots+\frac{x_0^{2p}}{(2p+1)!}A^{2p}, \qquad (8)$$

Then, (7) da (8) will be replaced by the following recurrent relations:

$$U_k=2U^2_{k-1}-I,\quad V_k=V_{k-1}U_{k-1},\quad k=1,2,...,n., \qquad (9)$$

Let us note that for $\cosh(lA)\approx U_n$ and $A^{-1}\sinh(lA)\approx V_n$ calculations we need to use just matrix multiplication. Stability of (9) scheme is shown, convergence theorem is proved and approximation error is estimated.